



Coincidence And Common Fixed Point Theorems In A G-Metric Space

K. Rauf and B.Y. Aiyetan

Department of Mathematics, University of Ilorin, Ilorin, Kwara State, Nigeria

ABSTRACT

Upon the introduction of metric space in 1906, there have being a lot of generalization of this space, like 2-metric space, D-metric space and so on. In 2006 the concept of G-metric space was established after proving that most properties of D-metric space are incorrect, which resulted to the establishment of more satisfactory properties of G-metric space and the relationship between metric space and G-metric space. In this paper, some theorems were proved to show the coincidence point and common fixed point in a G-metric space using some general contractive conditions formulated in G-metric form. Our results generalized and refine some results in the literatureperoxidation.

Keywords: G-metric, G- Cauchy, coincidence point, common fixed point, contractive conditions, G-continuous and commute

INTRODUCTION

Fixed point theory is a very important aspect of a nonlinear analysis. Finding solution to a nonlinear equation with the use of fixed point iteration has being a very interesting aspect and it has also being the focus of many researchers.

Mustafa and Sim (2006), introduced a new notion of generalized metric space called G-metric space, after proving that most of the results concerning the topological properties of D-metric space were incorrect. To take care of the setback, they gave a more appropriate notion of a generalized metrics, called G-metric space.

Abbas *et al.*, (2015) presented fixed point result of mappings satisfying a generalized rational contractive condition in the setup of multiplicative metric spaces. As an application, they obtained a common fixed point of a pair of weakly compatible mappings. Some common fixed point results of pair of rational contractive types mappings involved in cocyclic representation of a nonempty subset of a multiplicative metric space are also obtained. Shahzad. N. (2016). uses R-functions to present existence and uniqueness coincidence (common fixed) point results under a contractivity condition that extended some celebrated contractive mappings. In their main theorems, they employed a binary relation on the metric space, which does not have to be a partial order.

Khan (1981) proved some results on common fixed points for a pair of multivalued mappings defined on a closed subset of a complete metric space. In this paper we prove the coincidence (common fixed) point using some general contractive conditions in a G-metric space.

PRELIMINARIES

This section gives some definitions, theorems and lemmas used in this article.

Definition 2.1. (Mustafa and Sim (2006)). A G-metric space is a pair (X, G) where X is a nonempty set and $G : X \times X \times X \rightarrow [0, \infty)$ is a function such that, for all $x, y, z, a \in X$, the following conditions are satisfied

$$(G1) \quad G(x, y, z) = 0 \text{ if } x = y = z; \quad (2.1)$$

$$(G2) \quad G(x, x, y) > 0 \text{ for all } x, y \in X \text{ with } x \neq y; \quad (2.2)$$

$$(G3) \quad G(x, x, y) \leq G(x, y, z) \text{ for all } x, y, z \in X \text{ with } z \neq y; \quad (2.3)$$

$$(G4) \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots \text{ (symetry in all 3)}; \text{ and} \quad (2.4)$$

$$(G5) \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z). \quad (2.5)$$

In such a case, the function G is called a G-metric on X .

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Address Correspondence to:

krauf@unilorin.edu.ng

Definition 2.2. (Ravi, *et al.*, (2015)). Let (X, G) be a G -metric space, let $x \in X$ be a point and let $\{x_n\} \subseteq X$ be a sequence. We say that:

1. $\{x_n\}$ G -converges to x , and we write $\{x_n\} \rightarrow x$, if $\lim_{n,m \rightarrow \infty} G(x_n, x_m, x) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x) \leq \varepsilon$ for all $n, m \in \mathbb{N}$ such that $n, m \geq n_0$, in such case, x is the G -limit of $\{x_n\}$;

2. $\{x_n\}$ is G -Cauchy if $\lim_{n,m \rightarrow \infty} G(x_n, x_m, x_k) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x_k) \leq \varepsilon$ for all $n, m, k \in \mathbb{N}$ such that $n, m, k \geq n_0$; and

3. (X, G) is complete if every G -Cauchy sequence in X is G -convergent in X .

Definition 2.3 (Ravi, *et al.*, (2015)). Let (X, G) be a G -metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent:

1. $\{x_n\}$ G -convergent to x ;
2. $\lim_{n,m \rightarrow \infty} G(x_n, x_m, x_m) = 0$;
3. $\lim_{n,m \rightarrow \infty, m \geq n} G(x_n, x_m, x_n) = 0$;
4. $\lim_{n,m \rightarrow \infty, m > n} G(x_n, x_m, x_m) = 0$;
5. $\lim_{n,m \rightarrow \infty} G(x_n, x_n, x_m) = 0$;
6. $\lim_{n,m \rightarrow \infty, m \geq n} G(x_n, x_n, x_m) = 0$;
7. $\lim_{n,m \rightarrow \infty, m < n} G(x_n, x_n, x_m) = 0$; and
8. $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ and $\lim_{n,m \rightarrow \infty, m > n} G(x_n, x_{n+1}, x_m) = 0$.

Definition 2.4 (Ravi, *et al.*, (2015)). Let (X, G) be a G -metric space. We say that:

a. a mapping $T : X \rightarrow X$ is G -continuous at $x \in X$ if $\{Tx_m\} \rightarrow Tx$ for all sequence $\{x_n\} \subseteq X$ such that $\{x_m\} \rightarrow x$;

b. a mapping $F : X^n \rightarrow X$ is G -continuous at $(x_1, x_2, \dots, x_n) \in X^n$ if $\{F(x_1^m, x_2^m, \dots, x_n^m)\} \rightarrow F(x_1, x_2, \dots, x_n)$

for all sequence $\{(x_1^m, x_2^m, \dots, x_n^m)\} \subseteq X^n$ such that $\{x_i^m\} \rightarrow x_i$ for all $i \in \{1, 2, \dots, n\}$; and

c. a mapping $H : X^n \rightarrow X^m$ is G -continuous at $(x_1, x_2, \dots, x_n) \in X^n$ if $\pi_i^m \circ H : X^n \rightarrow X$ is G -continuous at (x_1, x_2, \dots, x_n) for all $i \in \{1, 2, \dots, m\}$, where $\pi_i^m : X^m \rightarrow X$ is the i th-projection of X^m

onto X (that is, $\pi_i^m(a_1, a_2, \dots, a_m) = a_i$ for all $(a_1, a_2, \dots, a_m) \in X^m$).

Theorem 2.1. (Ravi, *et al.*, (2015)). Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping such that there exists $\lambda \in [0, 1)$ satisfying

$$G(Tx, Ty, Tz) \leq \lambda G(x, y, z) \text{ for all } x, y, z \in X \tag{2.6}$$

Then, T has a unique fixed point. In fact T is a Picard operator.

Definition 2.5 (Ravi, *et al.*, (2015)). In the coincidence case, a sequence $\{x_n\}_{n \geq 0} \subseteq X$ is a Picard sequence of (T, g) if

$$gx_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}$$

If there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, so x_{n_0} is the coincidence point of T and g . Therefore, it is usual to assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 2.1 (Ravi, *et al.*, (2015)). If $T(X) \subseteq g(X)$, then there exists a Picard sequence of (T, g) based on any $x_0 \in X$.

Definition 2.6 There exists a real number $L \geq 0$ and a strict comparison function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $x, y \in X$

$$G(Tx, Ty, Ty) \leq LG(gx, Tgx, Tgx) + \varphi d(gx, gy, gy) \tag{2.7}$$

Definition 2.7 (Ravi, *et al.*, (2015)). There exists a monotone increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\varphi(0) = 0$ and a strict comparison function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for all $x, y \in X$

$$G(Tx, Tx, Ty) \leq \varphi G(x, x, Tx) + \psi G(x, x, y) \tag{2.8}$$

Definition 2.8 (Ravi, *et al.*, (2015)). There exist monotonically decreasing functions a, b, c from $(0, \infty)$ into $[0, 1)$ satisfying $a(t) + b(t) + c(t) < 1$ such that, for each $x, y \in X, x \neq y$.

$$G(Tx, Tx, Ty) \leq a(G(x, x, y))G(x, x, Tx) + b(G(x, x, y))G(y, y, Ty) + c(G(x, x, y))G(x, x, y) \tag{2.9}$$

Definition 2.9 (Ravi, *et al.*, (2015)). There exist nonnegative number a, b, c satisfying $a + b + c < 1$ such that, for each $x, y \in X$,

$$G(Tx, Tx, Ty) \leq aG(x, x, Tx) + bG(y, y, Ty) + cG(x, x, y) \tag{2.10}$$

RESULT

In this section, our main results were established by proving common fixed point theorems satisfying some contractive conditions in G -metric settings.

Theorem 3.1 Let (X, G) be a G -metric space and let $T, g : X \rightarrow X$ be two self-mappings satisfying (2.7). Assume that the following conditions are fulfilled.

- i. (X, G) is complete;
 - ii. $T(X) \subseteq g(X)$; and
 - iii. g is G -continuous and commutes with T .
- Then, T and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element but fixed in X . By assuming condition (i) and lemma 2.1 holds, then we can defined the sequence of iteration $\{gx_n\}$ in X as $gx_n = T^n x_0$ for all $n \geq 1$

If $gx_{n+2} = y_n = z_n$ and since (T, g) is a contractions satisfying (2.7) we have

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) = G(Tgx_n, Tgx_{n+1}, Tgx_{n+1}) \tag{3.1}$$

$$\begin{aligned} &\leq LG(gx_n, gx_{n+1}, gx_{n+1}) + \varphi G(gx_n, gx_{n+1}, gx_{n+1}) \\ &= LG(Tgx_{n-1}, Tgx_n, Tgx_n) + \varphi G(Tgx_{n-1}, Tgx_n, Tgx_n) \\ &\leq L^2 G(gx_{n-1}, gx_n, gx_n) + \varphi^2 G(gx_{n-1}, gx_n, gx_n) \end{aligned} \tag{3.2}$$

$$\begin{aligned} &= L^2 G(Tgx_{n-2}, Tgx_{n-1}, Tgx_{n-1}) \\ &\quad + \varphi^2 G(Tgx_{n-2}, Tgx_{n-1}, Tgx_{n-1}) \\ &\leq L^3 G(gx_{n-2}, gx_{n-1}, gx_{n-1}) \\ &\quad + \varphi^3 G(gx_{n-2}, gx_{n-1}, gx_{n-1}) \end{aligned}$$

Continue iteratively lead to

$$\leq L^n G(gx_1, gx_2, gx_2) + \varphi^n G(gx_1, gx_2, gx_2) \tag{3.3}$$

Hence

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \leq L^n G(gx_2, gx_1, gx_1) + \varphi^n G(gx_2, gx_1, gx_1)$$

and

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \leq [L^n + \varphi^n] G(gx_2, gx_1, gx_1) \tag{3.4}$$

If we let $k^n = [L^n + \varphi^n]$ for all $n \in \mathbb{N}$ where $0 < k^n < 1$ we have

$$\begin{aligned} G(gx_{n+1}, gx_{n+2}, gx_{n+2}) &\leq k^n G(gx_n, gx_{n+1}, gx_{n+1}) \\ &\leq k^n G(gx_0, gx_1, gx_1) \end{aligned} \tag{3.5}$$

as $k < 1$, so when $n \rightarrow \infty$, then

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \rightarrow 0$$

for all $n, m \in \mathbb{N}, m > n$, by repeated use of a rectangle inequality of G -metric, we have

$$\begin{aligned} G(gx_{n+1}, gx_{m+2}, gx_{m+2}) &\leq G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &\quad + G(gx_{n+2}, gx_{n+3}, gx_{n+3}) + \dots \\ &\quad + G(gx_{m+1}, gx_{m+2}, gx_{m+2}) \\ &\leq k^n G(gx_0, gx_1, gx_1) + k^{n+1} G(gx_0, gx_1, gx_1) + \dots \\ &\quad + k^{m-1} G(gx_0, gx_1, gx_1) \end{aligned}$$

$$\leq (k^n + k^{n+1} + \dots + k^{m-1}) G(gx_0, gx_1, gx_1)$$

by applying sum of geometric progression we have

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \leq \frac{k^n}{1-k} G(gx_0, gx_1, gx_1) \tag{3.6}$$

Since k^n value is positive and less than unity then

$$\lim G(gx_{n+1}, gx_{m+2}, gx_{m+2}) = 0 \text{ as } n, m \rightarrow \infty.$$

Therefore the sequence $\{gx_n\}$ is a G -Cauchy sequence.

We now show that p is the unique common fixed point of T and g . If contrary to our assumption, there exists another common fixed point $q \in X$ with $q \neq p$, from (2.7) we have that

$$G(p, q, q) \leq LG(p, Tp, Tp) + \varphi G(p, q, q) \tag{3.7}$$

this is a contradiction, which implies $p = q$.

Hence, the common fixed point of T and g exists and unique. ■

Theorem 3.2 Let (X, G) be a G -metric space and let $T, g : X \rightarrow X$ be two self-mappings satisfying (2.8). Assume that the following conditions are fulfilled:

- i. (X, G) is complete;
- ii. $T(X) \subseteq g(X)$; and
- iii. g is G -continuous and commutes with T .

Then, T and g have $p \in X$ such that $gp = Tp = p$.

Proof. Assume the conditions of Theorem 3.1 and since (T, g) is a contractions satisfying (2.8). If

$$G(gx_n, gx_n, gx_{n+1}) = G(Tgx_{n-1}, Tgx_{n-1}, Tgx_n) \quad G(p, q, q) \leq \varphi G(p, Tp, Tp) + \psi G(p, q, q)$$

3.8 3.14

then

$$\begin{aligned} G(gx_n, gx_n, gx_{n+1}) &\leq \varphi G(gx_{n-1}, gx_{n-1}, gx_n) + \psi G(gx_{n-1}, gx_{n-1}, gx_n) \\ &= \varphi G(Tgx_{n-2}, Tgx_{n-2}, Tgx_{n-1}) \\ &\quad + \psi G(Tgx_{n-2}, Tgx_{n-2}, Tgx_{n-1}) \\ &\leq \varphi^2 G(gx_{n-2}, gx_{n-2}, gx_{n-1}) + \\ &\quad \psi^2 G(gx_{n-2}, gx_{n-2}, gx_{n-1}) \quad 3.9 \\ &= \varphi^2 G(Tgx_{n-3}, Tgx_{n-3}, Tgx_{n-2}) \\ &\quad + \psi^2 G(Tgx_{n-3}, Tgx_{n-3}, Tgx_{n-2}) \\ &\leq \varphi^3 G(gx_{n-3}, gx_{n-3}, gx_{n-2}) \\ &\quad + \psi^3 G(gx_{n-3}, gx_{n-3}, gx_{n-2}) \\ &\vdots \\ &\leq \varphi^n G(gx_0, gx_0, gx_1) + \psi^n G(gx_0, gx_0, gx_1) \quad 3.10 \end{aligned}$$

Therefore

$$\begin{aligned} G(gx_n, gx_n, gx_{n+1}) &\leq \varphi^n G(gx_{n-1}, gx_{n-1}, gx_n) + \\ &\psi^n G(gx_{n-1}, gx_{n-1}, gx_n) \\ \text{and} \\ G(gx_n, gx_n, gx_{n+1}) &\leq \\ [\varphi^n + \psi^n] G(gx_n, gx_{n+1}, gx_{n+1}) \quad 3.11 \end{aligned}$$

assume $k^n = [\varphi^n + \psi^n]$ for all $n \in \mathbb{N}$ where $0 < k^n < 1$ we have

$$\begin{aligned} G(gx_n, gx_n, gx_{n+1}) &\leq k^n G(gx_{n-1}, gx_{n-1}, gx_n) \\ &\leq k^n G(gx_0, gx_1, gx_1) \quad 3.12 \end{aligned}$$

as $k < 1$, so when $n \rightarrow \infty$, then

$$G(gx_n, gx_n, gx_{n+1}) \rightarrow 0$$

For all $n, m \in \mathbb{N}, m > n$, hence

$$\begin{aligned} G(gx_n, gx_n, gx_m) &\leq G(gx_n, gx_n, gx_{n+1}) \\ &\quad + G(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\ &\quad + \dots \\ &\quad + G(gx_{m-1}, gx_{m-1}, gx_m) \\ &\leq k^n G(gx_0, gx_0, gx_1) \\ &\quad + k^{n+1} G(gx_0, gx_0, gx_1) + \dots \\ &\quad + k^{m-1} G(gx_0, gx_0, gx_1) \end{aligned}$$

$$\leq (k^n + k^{n+1} + \dots + k^{m-1}) G(gx_0, gx_0, gx_1)$$

which yields

$$G(gx_n, gx_n, gx_{n+1}) \leq \frac{k^n}{1-k} G(gx_0, gx_0, gx_1) \quad 3.13$$

$\lim G(gx_n, gx_n, gx_m) = 0$ as $n, m \rightarrow \infty$.

Therefore the sequence $\{gx_n\}$ is a G-Cauchy sequence.

We now show that p is the unique common fixed point of T and g . Let there exists another common fixed point $q \in X$ with $q \neq p$, from (2.8) we have that

This is a contradiction and $p = q$.

Hence, the common fixed point of T and g exists and unique. ■

Theorem 3.3 Let (X, G) be a G -metric space and let $T, g : X \rightarrow X$ be two self-mappings satisfying (2.9). Assume that conditions (i) - (iii) of Theorem 3.1 are satisfied. Then, T and g have a unique common fixed point, that is, there is a unique $p \in X$ such that $gp = Tp = p$.

Proof. The proof is similar to that of Theorem 3.2 ■

Theorem 3.4 Let (X, G) be a G -metric space and let $T, g : X \rightarrow X$ be two self-mappings satisfying (2.10). If conditions (i) - (iii) of Theorem 3.1 are valid. Then, T and g have $p \in X$ such that $gp = Tp = p$.

Proof. The proof is similar to that of Theorem 3.2 ■

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